HANF NUMBERS FOR FRAGMENTS OF $L_{\infty\omega}$

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ABSTRACT

We describe an ordinal $h(A)$ which plays a key role in the model theory of the admissible fragment L_A. In particular, the Hanf number of L_A is $\beth_{h(A)}$ If L_A is $L_{\kappa^+\omega}$ where $cf(k) > \omega$ then $h(A)$ can be characterized as the least ordinal which is not $H(\kappa^+)$-recursive.

0. Introduction

If L_A is an admissible fragment of $L_{\infty\omega}$, the usual examples showing that the Hanf number of L_A is large consist in defining a long well-ordering, $<$, of some type α, by a sentence $\phi(<)$ in L_A, and then concluding, by formalizing the cumulative hierarchy up to α, that the Hanf number of L_A is greater than $\beth_\alpha$. In §2 we prove that these are the only examples: if $h(A)$ is the least ordinal which cannot be "pinned down" by a sentence of L_A, then the Hanf number of L_A is precisely $\beth_{h(A)}$.

In §1, we give a precise definition of $h(A)$ and relate it to some concepts of a more recursion-theoretic nature. We also discuss some bounds on $h(A)$.

In §3, we show that $h(A)$ is also related to the two-cardinal problem for L_A.

In §4, we use the ideas of §§1, 2 to give a Cohen-style independence proof showing that the known bounds on the Hanf number, θ, of $L_{\kappa^+\omega}$ for regular κ ($\beth_{\kappa^+} < \theta < \beth_{(2^\kappa)^+}$) cannot be improved in ZFC.

1. L_A-accessible ordinals

We assume familiarity with lectures 13 and 14 of Keisler's book [6]. (The results in lectures 10, 15, 16 and 17 will fall out as corollaries of the results presented here.) The language $L_{\infty\omega}$ is just like the language $L_{\omega_1\omega}$ except that no bound is

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put on the sizes of the sets allowed in conjunctions and disjunctions. For any set A, L_A is $A \cap L_{\infty\omega}$. L_A is a *fragment* if it is closed under subformulas and the *finitary* logical operations (so the notion of fragment is very weak). L_A is an *admissible fragment* if A is an admissible set. By a *Skolem fragment* L_A we mean a fragment which contains for each formula $\exists x\phi(x, y_1, \cdots, y_n)$ a function symbol $f_{\exists x\phi}$. In Keisler [6] it is shown how to go from a fragment L_A to a Skolem fragment L_A^*. If L_A is a Skolem fragment, then T_{Skolem} is the set of sentences of L_A of the form

$$\forall y_1, \cdots, y_n[\exists x\phi(x, y_1, \cdots, y_n) \to \phi(f_{\exists x\phi}(y_1, \cdots, y_n),\ y_1, \cdots, y_n)].$$

By $H(\kappa)$, for an infinite cardinal κ, we mean $\{x: |TC(x)| < \kappa\}$ (where $TC(x)$ is the transitive closure of x). If $A = H(\kappa^+)$, then $L_A = L_{\kappa^+\omega}$.

1.1. Definition. Let L_A be an admissible fragment of $L_{\infty\omega}$. An ordinal α is *L_A-accessible* if there is a sentence ϕ of L_A involving the binary symbol $<$, the unary symbol U and possibly other non-logical symbols such that

i) in all models $\mathfrak{M}$ of ϕ, $U^{\mathfrak{M}}$ is the field of $<^{\mathfrak{M}}$ and $<^{\mathfrak{M}}$ well-orders $U^{\mathfrak{M}}$, and

ii) for some model $\mathfrak{M}$ of ϕ, $<^{\mathfrak{M}}$ has order type at least α.

We sometimes say that the sentence ϕ *pins down* the ordinal α.

By obvious coding and Löwenheim-Skolem arguments, certain modifications of Definition 1.1 would be equivalent to 1.1. For example, we could modify (ii) by requiring $<^{\mathfrak{M}}$ to have order type exactly α or modify (i) by requiring $U^{\mathfrak{M}} = \mathfrak{M}$ or both.

Note that every ordinal in A is L_A-accessible. Also, if $A \subseteq B$ then every L_A-accessible ordinal is L_B-accessible.

The L_A-accessible ordinals form an initial segment of the class *Ord* of ordinals. By results of Lopez-Escobar [8] this initial segment is proper. We give a different proof of this in Theorem 1.4.

1.2 Definition. Let $h(A)$ be the least ordinal not L_A-accessible.

We let $h(\kappa) = h(H(\kappa))$ so that $h(\kappa^+)$ is the least ordinal which is not $L_{\kappa^+\omega}$-accessible. From the remarks in the introduction it is clear that the Hanf number of L_A is at least $\beth_{h(A)}$. (By Theorem 2.1 it is exactly $\beth_{h(A)}$.) Hence, if one uses known upper bounds for the Hanf number of L_A one obtains as corollaries bounds for $h(A)$. For example, if $A \subseteq H(\kappa^+)$ then $h(A) < (2^\kappa)^+$, since the Hanf number of $L_{\kappa^+\omega}$ is $< \beth_{(2^\kappa)^+}$ (see Chang [4] or Lopez-Escobar [8]). This proof that $h(\kappa^+) < (2^\kappa)^+$ is unsatisfying on two counts. First, it is only an upper bound; and second, it involves the use of very large sets (sets of size $\beth_{2^\kappa}$) to prove a result about small sets (sets of size 2^κ). We now remedy both of these complaints. The result below, which has $h(\kappa^+) < (2^\kappa)^+$ as an immediate consequence, is proved directly. More formally, the proof can be carried out in Zermelo set theory with choice (and with suitable definitions of ordinals and the other concepts involved) whereas the proof via Hanf numbers cannot.

The basic tool used throughout this paper is that of a tree. The results obtained do not depend in a very sensitive way on the definition of tree, but for the sake of precision we define a *tree* to be a well-founded irreflexive (i.e. $x \nprec x$) relation $\langle \mathscr{T}, \prec \rangle$. We could also impose the conditions that $\prec$ be transitive and that for any $x \in \mathscr{T}$ the set $\{y \in \mathscr{T} : x \prec y\}$ is finite and linearly ordered by $\prec$, as will be clear from the proofs. This remark is intended to justify our use of the word "tree".

Every tree $\mathscr{T} = \langle \mathscr{T}, \prec \rangle$ has an associated rank function r mapping $\mathscr{T}$ into the ordinals. It is defined by

$$r(x) = \sup\{r(y) + 1 : y \prec x\}.$$

The ordinal of $\mathscr{T}$, $o(\mathscr{T})$, is defined by

$$o(\mathscr{T}) = \sup\{r(x) + 1 : x \in \mathscr{T}\}.$$

1.3 DEFINITION. An ordinal α is *tree accessible over A* if there is an $x \in A$ and a tree $\langle \mathscr{T}, \prec \rangle$ of subsets of x such that:

i) $\mathscr{T}$ and $\prec$ are Δ_0 definable (in the universe V) using parameters from A;

ii) $\alpha \leqq o(\mathscr{T})$.

1.4 THEOREM. *Given an admissible set A, an ordinal α is L_A-accessible just in case it is tree accessible over A.*

The easy part of the theorem is to show that every tree accessible ordinal α is L_A-accessible. Let x, $\mathscr{T}$, and $\prec$ be as in Definition 1.3. Let $\phi(y)$ be the Δ_0 formula defining $y \in \mathscr{T}$ and $\psi(y, z)$ the one defining $y \prec z$. Denote the parameters occuring in ϕ or ψ by $a_1, \cdots, a_n$, and let $w \in A$ be such that $x, a_1, \cdots, a_n \in w$ and w is transitive. Let $\theta \in L_A$ be the conjunction of the following sentences, where $\mathscr{T}, \prec, E, <$ and U are new relation symbols, f is a function symbol and $\bar{a}$ is a constant symbol, for each $a \in A$:

$$\forall v[\mathscr{T}(v) \to v \subseteq \bar{x}]$$

$$\bigwedge_{a \in w} \forall v[vE\bar{a} \leftrightarrow \bigvee_{b \in a} v = \bar{b}_n]$$

$$\forall v[\mathscr{T}(v) \leftrightarrow \phi(v, \bar{a}_1, \cdots, \bar{a}_n)]$$

$$\forall v, w[v \prec w \leftrightarrow \psi(v, w, \bar{a}_1, \cdots, \bar{a}_n]$$

"f maps $\mathscr{T}$ onto U"

"$<$ linearly orders U"

$\forall v, w[v \prec w \rightarrow f(v) < f(w)]$

$\forall w \forall y[(U(w) \wedge U(y) \wedge y < f(w)) \rightarrow \exists v \prec w(y \leqq f(v))]$

It is routine to check that every model $\mathfrak{M}$ of θ has $<^{\mathfrak{M}}$ well-ordering $U^{\mathfrak{M}}$ using the fact that ϕ, ψ are Δ_0 so that if $y \prec z$ holds in some model of θ then $y, z \subseteq x$ and $y \prec z$ holds in V. The second clause in 1.3 is obvious.

Before we get down to the hard half of Theorem 1.4, we state a well-known fact in the form of a lemma. It is this lemma which will allow us to conclude that certain trees of theories are Δ_0 definable.

1.5 LEMMA. *Let L_A be a Skolem fragment of $L_{\infty\omega}$, T a complete theory of L_A containing T_{Skolem}. Then T has a model iff it is closed under Hilbert-style provability.*

PROOF. By complete we mean of course that T contains exactly one of ϕ and $\neg\phi$ for each sentence $\phi \in L_A$. To say that T is closed under Hilbert style provability means that T contains all the L_A-axioms (1)–(4) and is closed under the L_A-rules (1)–(3) of Keisler [6], lecture 4. The proof, of course, is just like that for first order logic. One defines the Henkin model $\mathfrak{M}$ on the closed terms of L_A and proves that $\mathfrak{M} \models \phi$ iff $\phi \in T$, by induction on sentences ϕ of L_A.

PROOF OF 1.4. We may assume $\omega \in A$ since otherwise $L_A = L_{\omega\omega}$. Let ϕ be a sentence of L_A such that for all models $\mathfrak{M}$ of ϕ, $<^{\mathfrak{M}}$ well-orders $U^{\mathfrak{M}}$. Since $\phi \in A$ and A admissible there is a Skolem fragment $L_B \in A$ with $\phi \in L_B$. We let $L_B(c_1, \cdots, c_n)$ denote the fragment obtained from L_B by adjoining new constant symbols $c_1, \cdots, c_n$. Define $\mathscr{T}_n$ to be the set of all complete, consistent theories T of $L_B(c_1, \cdots, c_n)$ such that:

(i) $\phi \in T$;

(ii) $T_{\text{Skolem}} \subseteq T$;

(iii) $U(c_i) \in T$ (for $i = 1, \cdots, n$); and

(iv) $(c_{i+1} < c_i) \in T$ (for $i = 1, \cdots, n-1$).

Let $\mathscr{T} = \cup_{n<\omega} \mathscr{T}_n$. By Lemma 1.5, $\mathscr{T}$ is Δ_0 definable using parameters: the set of sentences of L_B, the set of L_B-axioms, the set of instances of L_B rules, ϕ, and T_{Skolem}. We make $\mathscr{T}$ into a tree $\langle \mathscr{T}, \prec \rangle$ by defining $T \prec T'$ iff T properly contains T'. Thus, if $T \prec T'$ then T involves more of the c_i's than T'. The tree $\mathscr{T}$ is well founded since a descending sequence

$$T_1 \succ T_2 \succ \cdots$$

could be used with Lemma 1.5 (applied to $\cup_n T_n$ and $\cup_n L_B(c_1 \cdots c_n)$) to get a model $\mathfrak{M}$ of $\cup_n T_n$ (and hence of ϕ) where $<^{\mathfrak{M}}$ is not wellordered (since each sentence $c_{n+1} < c_n$ is true in $\mathfrak{M}$). We define $r(T)$ for $T \in \mathscr{T}$ as above. A simple induction on $r(T)$ shows that if $\mathfrak{M} \models T$ and if $T \in \mathscr{T}_{n+1}$, then the predecessors of c_{n+1} under $<^{\mathfrak{M}}$ have order type $\leqq r(T)$, and if $T \in \mathscr{T}_0$ then $<^{\mathfrak{M}}$ has order type $\leqq r(T)$. Thus every model $\mathfrak{M}$ of ϕ has $<^{\mathfrak{M}}$ bounded in order type by $o(\mathscr{T})$, an ordinal which is tree accessible over A.

In the next two theorems we give a more concrete description of $h(A)$ for certain admissible sets A. An ordinal α is *A-recursive* if there is a tree $\langle \mathscr{T}, \prec \rangle$ where $\mathscr{T} \subseteq A$, $\mathscr{T}$ and $\prec$ are Δ_1 definable over $\langle A, \varepsilon \rangle$ (using parameters from A) and $\alpha = o(\mathscr{T})$. (If A is an initial segment L_β of the constructible universe then α is A-recursive just in case there is a β-recursive well-ordering of β of length α.) The A-recursive ordinals form a proper initial segment of the ordinals, but this segment is, in general, short of the ordinals of the next admissible set (in the sense of Barwise-Gandy-Moschovakis [3]). In particular, if $A = L_{\kappa^+}$ for a cardinal κ, then the least ordinal not A-recursive is far short of the least admissible ordinal $\tau > \kappa^+$. If $A = H(\kappa^+)$ then the least ordinal not A-recursive is less than $(2^\kappa)^+$.

1.6 THEOREM. *If $cf(\kappa) > \omega$ then $h(\kappa^+)$ is the least ordinal not $H(\kappa^+)$-recursive.*

PROOF. Since $y \subseteq x \in H(\kappa^+)$ implies $y \in H(\kappa^+)$, every tree accessible ordinal is $H(\kappa^+)$-recursive since the Δ_0 formulas which define a tree of subsets of x in V will define the same tree in $H(\kappa^+)$. Thus, by Theorem 1.4, $h(\kappa^+) \leqq \alpha$. This half of the proof works for any A satisfying: $y \subseteq x \in A$ implies $y \in A$.

Before proving that every $H(\kappa^+)$ recursive ordinal is L_A-accessible we need a definition and lemma which will also be used in section 4. Given functions f, g mapping κ into κ, we write $f \lhd g$ if and only if there is an $\alpha < \kappa$ such that $f(\beta) < g(\beta)$ whenever $\alpha < \beta < \kappa$. It is not difficult to see that each $\gamma < \kappa^+$ is the order type of a set $L \subseteq \kappa_\kappa$ well-ordered by $\lhd$.

1.7 LEMMA. *If $cf(\kappa) > \omega$ and $L \subseteq \kappa^\kappa$ is linearly ordered by $\lhd$ then L is well-ordered by $\lhd$.*

PROOF. Suppose that there was a descending sequence

$$f_0 \rhd f_1 \rhd \cdots \rhd f_n \rhd \cdots.$$

Let α_n be $< \kappa$ and such that $f_n(\beta) > f_{n+1}(\beta)$ for all $\beta > \alpha_n$. If $\alpha = \sup_n \alpha_n$ then $\alpha < \kappa$ (since $\mathrm{cf}(\kappa) > \omega$) and

$$f_0(\beta) > f_1(\beta) > \cdots > f_n(\beta) > \cdots$$

for $\alpha < \beta < \kappa$, which is a contradiction.

PROOF OF THEOREM 1.6. Assume that $cf(\kappa) > \omega$ so that Lemma 1.7 applies. Let $\langle \mathscr{T}, \prec \rangle$ be any $H(\kappa^+)$-recursive tree, let r be its rank function and let $\beta = o(\mathscr{T})$. We must show that β can be pinned down by some ϕ in $L_{\kappa^+\omega}$. Since $|\beta| \leqq |H(\kappa^+)|$ there is a well-ordered set $\langle U, < \rangle$ with $U \subseteq H(\kappa^+)$ of order type β and a function $f: \mathscr{T} \to U$ such that $f(x)$ is the $r(x)$th-element in the ordering $<$ (which isn't necessarily related to the ε relation on $H(\kappa^+)$). Since $\langle \mathscr{T}, \prec \rangle$ is $H(\kappa^+)$-recursive there are Σ_1 and Π_1 formulas $\sigma(x, y, a)$ and $\pi(x, y, b)$ such that both define the ordering $\prec$. The $a, b \in H(\kappa^+)$ are parameters. Let $X = TC(\{\kappa, a, b\})$ and note that $|X| = \kappa$. Let $\mathfrak{M}$ be the structure

$$\langle H(\kappa^+), \varepsilon, \prec, f, <, U, x \rangle_{x \in X}.$$

Let $\phi \in L_{\kappa^+\omega}$ be the conjunction of the following where, for $x \in X$, we use $\bar{x}$ as a constant symbol of $L_{\kappa^+\omega}$ and E is a symbol denoting ε:

i) all finite first order sentences true in $\mathfrak{M}$

ii) for each $x \in X$ the sentence

$$\forall v[vE\bar{x} \leftrightarrow \bigvee_{y \in x} v = \bar{y}].$$

The structure $\mathfrak{M}$ is a model of ϕ where $<^{\mathfrak{M}}$ has order type β so we need only check clause (i) of 1.1. Let

$$\mathfrak{M}_0 = \langle M_0, E_0, \prec_0, f_0, <_0, U_0, x_0 \rangle_{x \in X}$$

be any model of ϕ. By (2) we can assume that $x_0 = x$ for $x \in X$ and that $\langle \mathfrak{M}_0, E_0 \rangle$ is an end extension of $\langle X, \varepsilon \rangle$. We list some finite sentence which are true in $\mathfrak{M}$ and hence in $\mathfrak{M}_0$.

a) "Every ordinal has the type of some $\lhd$-linearly ordered set of functions mapping $\bar{\kappa}$ into $\bar{\kappa}$"

b) $\forall x[|x| \leqq \bar{\kappa}]$

c) $\forall x \forall y[x \prec y \leftrightarrow \sigma(x, y, \bar{a}) \leftrightarrow \pi(x, y, \bar{b})]$

d) "f maps $\prec$ onto $<$ so that $x \prec y$ implies $f(x) < f(y)$"

e) "If $x \prec y$ then there is no z such that for all $w \prec y, f(w) < z < f(y)$".

f) "For every x there is a rank function mapping the transitive closure of x onto an ordinal".

By (a) and Lemma 1.5 the ordinals of $\mathfrak{M}_0$ are all well-founded and hence, by (f), all sets in $\mathfrak{M}_0$ are well-founded. We may thus consider M_0 as a transitive set and $E_0 = \varepsilon \cap M_0^2$. By (b) we have $M_0 \subseteq H(\kappa^+)$. By (c) we have $\prec_0 = \prec \cap M_0^2$ so $\prec_0$ is a well-founded relation. By (d), $<_0$ must be a well ordering of U_0, so the proof is complete.

1.8 THEOREM. *Let $A \subseteq H(\omega_1)$ be admissible, let $\alpha = A \cap Ord$ and let ϕ be a sentence of L_A such that for each $\beta < \alpha$ there is a model $\mathfrak{M}$ of ϕ where $<^{\mathfrak{M}}$ has order type $\geqq \beta$. Then ϕ has a model where $<^{\mathfrak{M}}$ is not well ordered. Hence $h(A) = \alpha$.*

PROOF. We may suppose that A is the smallest admissible set with $\phi \in A$ so that A is countable and the compactness theorem holds for A-r.e. theories of L_A. We may also assume $\omega \in A$, since otherwise $\phi \in L_{\omega\omega}$ and the result is trivial.

Introduce the following new symbols: a binary relation symbol E, uinary relation symbols V and M, a function symbol f, a constant c and, for each $a \in A$, a constant symbol $\bar{a}$. In this proof and the next we use the notation $\psi^{V,E}$ to indicate that the sentence ψ from the language of set theory has had ε replaced by E and all the quantifiers relativized to V. Let T be the L_A theory containing:

(0) $\forall x[M(x) \vee V(x)]$

(i) $\phi^{(M)}$, the relativization of ϕ to M,

(ii) $\psi^{V,E}$ for each ψ in KP, the Kripke-Platek axioms for admissible sets

(iii) for each $a \in A$ the axiom $V(\bar{a}) \wedge \forall x[xE\bar{a} \leftrightarrow \bigvee_{b \in a} x = \bar{b}]$

(iv) $\neg \exists x[V(x) \wedge \bar{\phi} \in x \wedge x$ is admissible$]$

(v) for each $\beta \in A$ the axiom $\bar{\beta}Ec$ and $(c$ is an ordinal$)^{V,E}$

(vi) "f maps the predecessors of c into U so that xEy implies $f(x) < f(y)$".

For $T_0 \subseteq T$, $T_0 \in A$ let $\gamma \in A$ be greater than all ordinal mentioned in T_0. Then T_0 will have a model of the form

$$\langle A \cup M;\, A, \varepsilon_A, \gamma; M, <, U \cdots; f\rangle.$$

By compactness, there is a model

$$\langle A' \cup M;\, A', E', c;\, M, <, U, \cdots; f\rangle$$

of all of T. By (iii) and (v), $\langle A', E\rangle$ is a proper end extension of $\langle A, \varepsilon\rangle$ and, by (ii) and (iv), the predecessors of c are not wellfounded (otherwise $L_\alpha(\phi)$ is an "element" of (A'). Thus, by (i) and (vi), $\langle \mathfrak{M}, <, U, \cdots\rangle$ is a model of ϕ with $<$ not well-founded.

Actually, in the model $\langle M, <, U, \cdots\rangle$, $<$ must contain a copy of the rationals, since the predecessors of c do in $\langle A', E\rangle$, so we see that the above also gives us a proof of Theorem 12 in Keisler [6]. Theorem 1.4 appears as Theorem 2.16 in Barwise [2], but the proof presented here is neater.

In the next section we need to know that $h(A)$ is always closed under ordinal addition and multiplication. Actually, much more is true. For terminology see Jensen-Karp [5].

1.9 THEOREM. *For any A, $h(A)$ is primitive recursively closed.*

Note that one cannot prove in ZFC that $h(A)$ is admissible since if $cf(\kappa) > \omega$ and $V = L$ then $h(\kappa^+)$ is not admissible by the remarks following Theorem 1.4.

To prove 1.9 let $F : \mathrm{Ord} \to \mathrm{Ord}$ be a primitive recursive function of one variable. The more general case where F has n arguments is handled similarly. There are Σ_1 and Π_1 formulas $\sigma(x, y)$ and $\pi(x, y)$ of set theory such that one can prove in KP

$$\forall\alpha\exists!\beta\sigma(\alpha,\beta) \wedge \forall\alpha,\beta[\sigma(\alpha,\beta)\leftrightarrow\pi(\alpha,\beta)] \ldots\ldots (*)$$

and such that σ defines F, that is, for all α, $\sigma(\alpha, F(\alpha))$ holds. We wish to show that if α is A-accessible, so is $F(\alpha)$. Let $\phi = \phi(<, U)$ be as in 1.1. As remarked there, we can assume that some model $\mathfrak{M}$ of ϕ has $<^{\mathfrak{M}}$ or order exactly α. We add new symbols to L as follows: $<'$, U' (to define $F(\alpha)$ in the sense of 1.1); V, E, M, g, c, d. Let ψ be the conjunction of:

0) $\forall x(V(x) \vee M(x))$

i) "c and d are ordinals" (in the sense of $\langle V, E\rangle$)

ii) "g maps $<$ 1-1 onto the E-predecessors of c in an order preserving manner"

iii) $\sigma(c, d)^{V,E}$

iv) $KP^{V,E}$

v) $\phi(<, U)$

vi) "$U' = \{x\colon V(x) \wedge xEd\}$ and $<' = E \cap U'$,"

It is clear that there is a model $\mathfrak{N}$ of ψ with $(<')^{\mathfrak{N}}$ a well-ordering of $(U')^{\mathfrak{N}}$ of order type $F(\alpha)$. We need to check condition (i) of 1.1. Let

$$\langle M \cup V;\ V, E, c, d, U', <';\ M, <, U, \cdots;\ g\rangle$$

be any model of ψ. We need to see that $<'$ is a well-ordering. We identify the well-founded part of $\langle V, E\rangle$ with a transitive set B, we let $\beta = B \cap \mathrm{Ord}$ and we let C be the set of sets constructible before β. By Lemma 3.3 of Barwise [2], C is admissible. (Actually, all of B is admissible.) By (v), (i) and (ii), $c < \beta$ so c is a standard ordinal γ. Since C is admissible it is closed under F so $F(\gamma) < \beta$. We need to see that $F(\gamma) = d$. But $\pi(\gamma, F(\gamma))$ is true so is true in C (since π is Π_1) and hence $\sigma(\gamma, F(\gamma))$ is true in C by (*) so $\sigma(\gamma, F(\gamma))$ holds in $\langle V, E\rangle$ (since σ is Σ_1). Thus, by (iii) and (*), $d = F(\gamma)$ and $<'$ is consequently a well-ordering, by (vi).

Similarly, using the fact that the enumerating functions of the primitively recursively closed ordinals is provably Δ_1 in KP, one sees that $h(A)$ is a fixed point of primitive recursively closed ordinals.

2. Hanf numbers and accessible ordinals

Given a cardinal ρ, we define $\beth_0(\rho) = \rho$ and for $\alpha > 0$, $\beth_\alpha(\rho) = \sup_{\beta<\alpha}(2^{\beth_\beta(\rho)})$ We write $\beth_\alpha$ for $\beth_\alpha(0)$. Thus $\beth_\omega = \omega$ and $\beth_{\omega+1} = 2^\omega$. In general, $\beth_\alpha = |V_\alpha|$ where $V_\alpha = \bigcup_{\beta<\alpha} P(V_\beta)$.

2.1 THEOREM. *Let L_A be an admissible fragment. The Hanf number of L_A is $\beth_{h(A)}$. More precisely,*

1) if $\alpha < h(A)$ then there is a sentence ϕ of L_A with models of power $\beth_\alpha$ but not arbitrarily large models, and

2) if ϕ is a sentence of L_A having models of power $\geqq \beth_\alpha$ for all $\alpha < h(A)$ then ϕ has arbitrarily large models.

The proof of (1) was sketched in the introduction. The proof of (2) requires some work.

Let us fix, for Lemma 2.2, a Skolem fragment L_B (not necessarily admissible) of $L_{\infty\omega}$.

Given a structure $\mathfrak{M}$ for L_B and an integer $n \geqq 0$, a set of *n-variable indiscernibles* in $\mathfrak{M}$ for L_B is a linearly ordered set $\langle X, <\rangle$ with $X \subseteq M$ such that for any two increasing sequences from X, of length n, say $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$, we have

$$(\mathfrak{M}, x_1 \cdots x_n) \equiv_{L_B} (\mathfrak{M}, y_1 \cdots y_n).$$

If $\langle X, <\rangle$ is a set of n-variable indiscernibles for every $n \geqq 0$ then we say that $\langle X, <\rangle$ is a set of indiscernibles in M for L_B.

2.2 LEMMA. *Suppose that for each $n \geqq 0$, $\mathfrak{M}_n$ is a model of T_{Skolem} and $\langle X_n, <_n\rangle$ is an infinite set of n-variable indiscernibles in $\mathfrak{M}_n$. Suppose further that for all $x_1 < \cdots < x_n$ in $\langle X_n, <_n\rangle$ and all $y_1 < \cdots < y_n$ in $\langle X_m, <_m\rangle$ where $m > n$ we have*

$$(\mathfrak{M}_n, x_1 \cdots x_n) \equiv_{L_B} (\mathfrak{M}_m, y_1 \cdots y_n).$$

Given any infinite linearly ordered set $\langle Y, <\rangle$ there is a model $\mathfrak{N}$ such that

$\langle Y, < \rangle$ is a set of indiscernibles in $\mathfrak{N}$ and $x_1 < \cdots < x_n$ in $\langle X_n, <_n \rangle$ and $y_1 < \cdots < y_n$ in $\langle Y, < \rangle$ implies

$$(\mathfrak{M}_n, x_1 \cdots x_n) \equiv_{L_B} (\mathfrak{N}, y_1 \cdots y_n).$$

In particular, $\mathfrak{M}_n \equiv_{L_B} \mathfrak{N}$ for all n.

PROOF. Let $L_C = \cup_n L_B(c_1 \cdots c_n)$ and let T be the set of sentences $\phi(c_1 \cdots c_n)$ of L_C such that $\phi(c_1 \cdots c_n)$ is true in M_n for some (hence every) interpretation of $c_1 \cdots c_n$ by an n-triple from $\langle X_n, <_n \rangle$. Then, by Lemma 1.5, T is a consistent theory and so has a model $(\mathfrak{N}_0, d_1 \cdots d_n \cdots)$. The set of d's is an infinite set of indiscernibles in $\mathfrak{N}$ (ordered by $d_i < d_j$ iff $i < j$). The conclusion of the lemma then follows from the Stretching Theorem (Theorem 19 (ii)) of Keisler [6].

PROOF OF THEOREM 2.1(ii). We may assume $\omega \in A$. Let ϕ be a sentence of L_A which does not have arbitrarily large models. We wish to show that the cardinality of the models of ϕ is bounded by $\beth_\alpha$ for some $\alpha < h(A)$. The proof parallels the proof of Theorem 1.4. Let $L_B \in A$ be a Skolem fragment with $\phi \in L_B$ and let T_{Skolem} be the corresponding set of Skolem axioms. For $0 \leqq n < \omega$ let $\mathscr{T}_n$ be the set of all complete consistent theories T of $L_B(c_1 \cdots c_n, X, <)$ such that:

i) $\phi \in T$;

ii) $T_{\text{Skolem}} \subset T$;

iii) "$<$ linearly orders the infinite set X" $\in T$;

iv) "$\forall x_1 \cdots x_n \in X[x_1 < \cdots < x_n \rightarrow (\phi(x_1 \cdots x_n) \leftrightarrow \phi(c_1 \cdots c_n))]$" $\in T$ for all formulas $\phi(x_1 \cdots x_n)$ of L_B;

v) "$c_1 \cdots c_n \in X$ and $c_1 < \cdots < c_n$" $\in T$.

Let $\mathscr{T} = \cup_n \mathscr{T}_n$. Again by Lemma 1.5, $\mathscr{T}$ is Δ_0 definable using parameters in A. We define $T \prec T'$, where $T \in \mathfrak{I}_n$ and $T' \in \mathfrak{I}_m$, by: $T \cap L_B(c_1 \cdots c_n)$ properly contains $T' \cap L_B(c_1 \cdots c_m)$.

We claim that $\langle \mathscr{T}, \prec \rangle$ is well-founded. Otherwise let

$$T_1 \succ T_2 \succ \cdots$$

be a descending chain in $\mathscr{T}$, and let $T = \cup_n T_n$. By Lemma 2.2 T has arbitrarily large models N which are hence models of ϕ, contradicting our assumption that the models of ϕ are bounded in cardinality. Define $r(T)$ for $T \in \mathscr{T}$ as before. Let us assume, for the moment, the following lemma, and show how it allows us to conclude the proof of 2.1.

2.3 LEMMA. *If $T \in \mathscr{T}_n$ and $(\mathfrak{M}, X, <, c_1 \cdots c_n) \models T$ then $|X| < \beth_{\omega(\alpha+1)}(\kappa)$, where $\alpha = r(T)$ and $\kappa = 2^{|B|}$.*

In particular, if $T \in \mathscr{T}_0$, then the lemma says that $|M| \leqq \beth_{\omega(\alpha+1)}(\kappa)$ where $\alpha = r(T)$, for all models M of T. But every model $\mathfrak{M}$ of ϕ can be expanded to a model of some $T \in \mathscr{T}_0$, so every model $\mathfrak{M}$ of ϕ has cardinality $\leqq \beth_{\omega(\alpha+1)}(\kappa)$, where $\alpha = o(\mathscr{T})$ and $\kappa = 2^{|B|}$. Now $B \in A$ so $r(B) \in A$ so $|B| \leqq \beth_\beta$ for some $\beta \in A$. Thus $\kappa \leqq \beth_{\beta+1} = \beth_\gamma$ where $\gamma \in A$ and $\beth_{\omega(\alpha+1)}(\kappa) \leqq \beth_{\omega(\alpha+1)}(\beth_\gamma) = \beth_{\gamma+\omega(\alpha+1)}$. Now α is tree accessible over A since $\alpha = o(\mathscr{T})$, so is it L_A-accessible by Theorem 1.4. But ω, γ and 1 are also L_A-accessible so the ordinal $\gamma + \omega(\alpha + 1)$ is also L_A-accessible by Theorem 1.9.

To conclude the proof we must return and prove Lemma 2.3. This is done by induction on $\alpha = r(T)$. Suppose that $|X| \geqq \beth_{\omega(\alpha+1)}(\kappa)$. For each increasing sequence $\vec{x} = x_1 \cdots x_{n+1}$ from $\langle X, < \rangle$ let $T_{\vec{x}}$ be the complete L_B theory of $(\mathfrak{M}, X, <, x_1 \cdots x_{n+1})$. Now, since $T_{\vec{x}} \subseteq L_B$ there are at most $2^{|B|} = \kappa$ such $T_{\vec{x}}$. By the Erdös-Rado Theorem there is a subset $X_0 \subseteq X$ of cardinality $> \beth_{\omega\alpha}(\kappa)$ such that $T_{\vec{x}} = T_{\vec{y}}$ for all increasing $(n+1)$-triples $\vec{x}$ and $\vec{y}$ from $\langle X_0, < \rangle$. Call this theory T'. Then $T' \prec T$ so $\alpha' = r(T') < \alpha$. But $(\mathfrak{M}, X_0, <, x_1 \cdots x_{n+1})$ and T' violate the inductive hypothesis since $|X_0| > \beth_{\omega\alpha}(\kappa) \geqq \beth_{\omega(\alpha'+1)}(\kappa)$. This concludes the proof of the lemma and hence the theorem.

As a consequence of Theorem 2.1 and the results of the previous section, we have a precise description of the Hanf number for single sentences of L_A. If A is countable and admissible then one can handle A-r.e. theories, so one might argue that it is the Hanf number of A-r.e. theories which is interesting. The next result shows that it is also $\beth_\alpha$ where $\alpha = A \cap \mathrm{Ord}$.

2.4 THEOREM. *Let L_A be a countable admissible fragment and let T be an A-r.e. theory. If every $T_0 \subseteq T$ with $T_0 \in A$ has arbitrarily large models then so does T.*

PROOF. Of course, since the downward Löwenheim Skolem theorem holds for L_A, T will have models of all infinite cardinalities. We can assume that L_A is a Skolem fragment and that $T_{\mathrm{Skolem}} \subseteq T$. Let T' be the following A-recursive set of sentences

$$c_i \neq c_j \text{ for } 0 \leqq i < j < \omega$$

$$\phi(c_{i_1} \cdots c_{i_n}) \leftrightarrow \phi(c_{j_1} \cdots c_{j_n}) \text{ for all } i_1 < \cdots < i_n,\ j_1 < \cdots < j_n \text{ and } \phi(x_1 \cdots x_n) \in L_A.$$

Here $C = \{c_1 \cdots c_n \cdots\}$ is a set of new constant symbols, $C \in A$. Now T has arbitrarily large models iff $T \cup T'$ is consistent. Thus, if $T \cup T'$ is not consistent there is some $T_0 \subseteq T$, $T_0 \in A$ such that $T_0 \cup T'$ is not consistent so T_0 does not have arbitrarily large models. This proof works, of course, for any L_A which is Σ_1-compact.

3. Morley's two cardinal theorem

We give one final example to show that the ordinal $h(A)$ plays a key role in the model theory of L_A. A structure $\mathfrak{M} = \langle M, U, \cdots \rangle$ is *a model of type* (κ, λ) if $|M| = \kappa$ and $|U| = \lambda$. A sentence $\phi(U)$ *admits* (κ, λ) if ϕ has a model of type (κ, λ). See lecture 17 of Keisler [6].

3.1 THEOREM. *Let ϕ be a sentence of $L_{\infty\omega}$ and let L_A be the least admissible fragment with $\phi \in L_A$. Suppose that for all $\alpha < h(A)$ there is a $\lambda \geqq |A|$ such that ϕ admits $(\beth_\alpha(\lambda), \lambda)$. Then for all $\kappa \geqq |A|$ ϕ admits $(\kappa, |A|)$.*

PROOF. We may assume that in all models M of ϕ, $|U| \geqq |A|$.

Proceed exactly as in the proof of Theorem 2.1, but modify the definition of $\mathscr{T}_n$ by adding clause

vi) "$\forall x_1 \cdots x_n\, y_1 \cdots y_n \in X(x_1 < \cdots < x_n \wedge y_1 < \cdots < y_n \wedge f(x_1 \cdots x_n) \in U \rightarrow f(x_1 \cdots x_n) = f(y_1 \cdots y_n))$" $\in T$ for each n-place function symbol f of L_B.

Now if $\mathscr{T}$ is not well-founded and $T_1 \succ T_2 \succ \cdots$ is a path through $\mathscr{T}$, then, as in Lemma 2.2, the stretching theorem applied to $T = \cup_n T_n$ yields models of type $(\kappa, |A|)$ for all $\kappa \geqq |A|$. On the other hand, if $\mathscr{T}$ is well-founded, then, by induction on $\mathscr{T}$ as in Lemma 2.3, one can show that if $\alpha = o\ (\mathscr{T})$, ϕ has no models of type (κ, λ) when $\kappa \geqq \beth_{\omega.(\alpha+1)}(\lambda)$.

4. An independence proof

In §1, we saw that, for $cf(\kappa) > \omega$, $\kappa^+ < h(\kappa^+) < (2^\kappa)^+$. If $2^\kappa \gg \kappa^+$, this is not a very good estimate for $h(\kappa^+)$. Unfortunately, one cannot in general do much better using just the axioms *ZFC*, as we show in this section. For a regular cardinal κ, we shall describe two different Cohen extensions which make 2^κ large. The two extensions will, in fact, have identical cardinal exponentiation. However, in the first, $h(\kappa^+)$ will be less than κ^{++}, whereas in the second $h(\kappa^+)$ will become greater than 2^κ. Theorem 4.2 states our result precisely in terms of Boolean-valued models. This theorem will have, as immediate corollaries, various independence results like, for example,

4.1 COROLLARY. *If ZFC is consistent, so are:*

1) $ZFC + 2^{\omega} = \omega_1 + 2^{\omega_1} = \omega_{84} + h(\omega_2) < \omega_3$.

2) $ZFC + 2^{\omega} = \omega_1 + 2^{\omega_1} = \omega_{84} + h(\omega_2) > \omega_{84}$.

See Rosser [10] for basic material on Boolean-valued models. For simplicity we always assume GCH in the ground model, V; this is clearly permissible from the point of view of obtaining independence proofs. If κ and θ are regular cardinals with $\omega \leqq \kappa < \theta$, we say that the complete Boolean algebra, $\mathscr{B}$, is a *minimal* (κ, θ) *extension* iff $\mathscr{B}$ preserves cardinals and cofinalities and, for all infinite cardinals $\lambda < \kappa$,

$$[\![2^{\lambda} = \lambda^{+}]\!]^{\mathscr{B}} = 1,$$

while for all cardinals $\lambda \geqq \kappa$,

$$[\![2^{\lambda} = \max(\lambda^{+}, \theta)]\!]^{\mathscr{B}} = 1.$$

These conditions determine all cardinal exponentiations in $V^{\mathscr{B}}$, and these exponentiations are as small as possible consistent with 2^{κ} being blown up to θ.

4.2 THEOREM. *Assume the GCH, and let $\omega < \kappa < \theta$, where κ and θ are regular. Then there are complete Boolean algebras $\mathscr{B}$ and $\mathscr{C}$ such that $\mathscr{B}$ and $\mathscr{C}$ are both minimal (κ, θ) extensions, but*

1) In $V^{\mathscr{B}}$, $[h(\kappa^{+}) < \kappa^{++}]^{\mathscr{B}} = 1$.

2) In $V^{\mathscr{C}}$, $[h(\kappa^{+}) > \theta]^{\mathscr{C}} = 1$.

PROOF. $\mathscr{B}$ will be the standard Boolean algebra for making $2^{\kappa} = \theta$. Let $(2^{\theta})_{\kappa}$ be the topological space 2^{θ} with the κ-topology—i.e., basic neighborhoods are determined on $< \kappa$ coordinates. Let $\mathscr{B} = \text{r.o.}((2^{\theta})_{\kappa})$—i.e., the regular-open algebra of $(2^{\theta})_{\kappa}$. That $\mathscr{B}$ is a minimal (κ, θ) extension is well-known, so we need only check (1).

Let $\mathscr{A} = \text{r.o.}((2^{(\kappa^{+})})_{\kappa})$. GCH holds in $V^{\mathscr{A}}$, so $[h(\kappa^{+}) < \kappa^{++}]^{\mathscr{A}} = 1$. By a symmetry argument, for each ordinal α, $[h(\kappa^{+}) = \alpha]^{\mathscr{A}}$ is 0 or 1 (since this Boolean value is fixed by all authomorphisms of A). Thus, for some $\alpha < \kappa^{++}$, $[\![h(\kappa^{+}) = \alpha]\!]^{\mathscr{A}} = 1$. We shall now show that $[h(\kappa^{+}) \leqq \alpha]^{\mathscr{B}} = 1$ (actually, equality holds in this last expression, but we do not need that here).

Suppose $[\![h(\kappa^{+}) \leqq \alpha]\!]^{\mathscr{B}} < 1$. Then, by a symmetry argument applied to $\mathscr{B}$, $[\![h(\kappa^{+}) > \alpha]\!]^{\mathscr{B}} = 1$. Thus, there is a $\phi \in V^{\mathscr{B}}$ such that $[\![\phi \in L_{\kappa^{+}\omega}]\!]^{\mathscr{B}} = 1$ and $[\![\phi$ pins down $\alpha]\!]^{\mathscr{B}} = 1$ (see Definition 1.1). We consider $\mathscr{A}$ to be a sub-algebra of $\mathscr{B}$ by identifying elements of $\mathscr{A}$ with cylinders over the first κ^{+} coordinates in $\mathscr{B}$; so $V^{\mathscr{A}} \subseteq V^{\mathscr{A}}$. By symmetry, we may assume $\phi \in V^{\mathscr{A}}$ (if not, since ϕ depends on only κ coordinates in $\mathscr{B}$, we may simply apply an authomorphism of $\mathscr{B}$ to move ϕ into

$V^{\mathscr{A}}$). Now (i) of Definition 1.1 holds in V^A since it is a universal statement. By (ii), let $\mathfrak{M} \in V^{\mathscr{B}}$ be such that it is $\mathscr{B}$-valid that $\mathfrak{M} \models \phi$ and that in $\mathfrak{M}, <^{\mathfrak{M}}$ has order type $\geqq \alpha$. We may also assume that $[\![|M| = \kappa^+]\!]^{\mathscr{B}} = 1$ by the Löwenheim-Skolem theorem. Hence, again by symmetry, we may assume $\mathfrak{M} \in V^A$ (if not, since $\mathfrak{M}$ depends on only κ^+ coordinates, we may apply an authomorphism to $\mathscr{B}$ which fixes ϕ and moves $\mathfrak{M}$ into $V^{\mathscr{A}}$). Thus, in $V^{\mathscr{A}}$, α is $L_{\kappa^+\omega}$ accessible, a contradiction.

We complete the proof of Theorem 3.2 by describing the algebra $\mathscr{C}$. Recall the ordering $\lhd$ on κ^κ defined in §1. In $L_{\kappa^+\omega}$, one can write a sentence whose models are $\lhd$-linearly ordered (and hence, by Lemma 1.7, well-ordered) subsets of κ^κ. Thus, if $\mathscr{C}$ adds a generic $\lhd$-well-ordered sequence of type θ, θ will become $L_{\kappa^+\omega}$-accessible in $V^{\mathscr{C}}$.

It is easiest to describe $\mathscr{C}$ in terms of a notion of forcing (i.e., partial ordering), $\langle P, \prec \rangle$. Then $\mathscr{C}$ will be the unique complete Boolean algebra containing $\langle P, \prec \rangle$ as a dense subset.

Intuitively, $\langle P, \prec \rangle$ is to add a function $F\colon \theta \times \kappa \to \kappa$. If we define $f_\gamma \in \kappa^\kappa$ by $f_\gamma(\xi) = F(\gamma, \xi)$, then the sequence $\langle f_\gamma \colon \gamma < \theta \rangle$ is to be $\lhd$-increasing.

More precisely we let $P = \cup \{P_{s,\eta} \colon s \subseteq \theta \wedge |s| < \kappa \wedge \eta < \kappa\}$, where

$$P_{s,\eta} = \{\langle s, \eta, g \rangle \colon g \in \kappa^{(s \times \eta)}\}.$$

If $p = \langle s, \eta, g \rangle \in P_{s,\eta}$, p is interpreted as defining $f_\alpha(\xi)$ to be $g(\alpha, \xi)$ for $\alpha \in s$ and $\xi < \eta$; in addition, p "says" that for $\xi \geqq \eta$, $f_\alpha(\xi) < f_\beta(\xi)$ whenever $\alpha < \beta$ and $\alpha, \beta \in s$. We thus define $\langle s', \eta', g' \rangle \preccurlyeq \langle s, \eta, g \rangle$ iff $s' \supseteq s$, $\eta' \geqq \eta$, $g' \restriction (s \times \eta) = g$, and, whenever $\alpha < \beta$ and $\alpha, \beta \in s$, $g'(\alpha, \xi) < g'(\beta, \xi)$ for all ξ such that $\eta \leqq \xi < \eta'$.

That $\mathscr{C}$ is a minimal (κ, θ) extension is standard. Also, in $V^{\mathscr{C}}$ there is a $\lhd$-well-ordered set of type θ, so $[h(\kappa^+) > \theta]^{\mathscr{C}} = 1$.

5. Historical remarks

The first Hanf number computation which really got to the heart of the matter was Morley's (unpublished) proof that the Hanf number for ω-logic is $\beth_\alpha$ where α is the least non-recursive ordinal (see Morley [9]). This result is a corollary of 2.1 and 1.8. This was generalized in Barwise [1], where it was shown that for admissible $A \subseteq H(\omega_1)$, the Hanf number of L_A is $\beth_\alpha$ where $\alpha = A \cap \mathrm{Ord}$. This is also a corollary of 2.1 and 1.8. Theorem 2.1 was obtained independently by Morley (in the context of omitting types) and the authors in late 1967. The existence of Keisler [6] makes the task of writing up the result much easier, and in part accounts (in conjunction with the emergence of 1.4) for its appearance at this time. Theorem

4.2 first appeared in Kunen [7], and has also become easier to write up due to the recent literature on Boolean-valued models (see, e.g., Rosser [10] and Scott-Solovay [11]). For more history, see Chang [4].

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